8. ALISHAEV M.G., BERNADINER M.G. and ENTOV V.M., Effect of the limit gradient on the loss of oil displaced by water. In: Problems of Non-linear Filtration and of Oil and Gas Yield in the Exploitation of Oil and Gas Deposits. Moscow, Izd-e In-ta geologii i razrab. goryuchikh iskopaemykh. 1972.
9. ENTROV V.M., MALAKHOVA T.A., PANKOV V.N. and PAN'KO S.V., Calculation of the limit equilibrium of retained viscoplastic oil extracted from a non-uniform stratified layer by water. PMM Vol.44, No.1, 1980.

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ON THE JOINT APPLICATION OF CARTESIAN AND BIPOLAR COORDINATES TO SOLVE BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY AND ELASTICITY THEORY*

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Equations are obtained that connect harmonic functions with separated variables in Cartesian and bipolar coordinates. These equations can be used to investigate a number of new boundary value problems of potential theory and elasticity theory for domains bounded by Cartesian and bipolar coordinate system coordinate lines.

1. Consider a plane domain whose boundary is formed by two intersecting circles. The solution of internal boundary value problems for such domains (circular cresents) is found in bipolar coordinates α, β defined by the relations $(a > 0)$ [1]

$$x = \frac{a \operatorname{sh} \alpha}{\operatorname{ch} \alpha + \cos \beta}, \quad y = \frac{a \sin \beta}{\operatorname{ch} \alpha + \cos \beta} \; (-\infty < \alpha < \infty, -\pi \leqslant \beta \leqslant \pi) \tag{1.1}$$

The arcs of the circles forming the circular cresent are the coordinate lines $\beta = \text{const}$, and pass through the point $x = \pm a, y = 0$. The quantity β is measured by the angle between the tangent to the arc at the point $x = a$, $y = 0$ and the segment $(-a, a)$ of the x axis corresponding to the value $\beta = 0$. Within the domain under consideration the coordinate α varies between the limits $-\infty$ and ∞. Particular solutions of the Laplace equation in bipolar coordinates, obtained by separation of variables and bounded as $\alpha \to \pm\infty$, have the following form

$$\cos \lambda\alpha \left\| \begin{matrix} \operatorname{ch} \lambda\beta \\ \operatorname{sh} \lambda\beta \end{matrix} \right\|, \quad \sin \lambda\alpha \left\| \begin{matrix} \operatorname{ch} \lambda\beta \\ \operatorname{sh} \lambda\beta \end{matrix} \right\| \quad (-\infty < \lambda < \infty)$$

Theorem 1. The following equations hold for $-\pi < \beta < \pi$

$$\operatorname{sh} \lambda y \left\| \begin{matrix} \cos \lambda x \\ \sin \lambda x \end{matrix} \right\| = \int_{-\infty}^{\infty} C(\lambda, \tau) \operatorname{sh} \tau\beta \left\| \begin{matrix} \cos \tau\alpha \\ \sin \tau\alpha \end{matrix} \right\| d\tau \tag{1.2}$$

$$\operatorname{ch} \lambda y \left\| \begin{matrix} \cos \lambda x \\ \sin \lambda x \end{matrix} \right\| - \left\| \begin{matrix} \cos \lambda a \\ 0 \end{matrix} \right\| = \int_{-\infty}^{\infty} C(\lambda, \tau) \operatorname{ch} \tau\beta \left\| \begin{matrix} \cos \tau\alpha \\ \sin \tau\alpha \end{matrix} \right\| d\tau$$

$$C(\lambda, \tau) = \frac{\lambda a}{\operatorname{sh} \pi\tau} e^{-i\lambda a} \Phi(1 - i\tau, 2; 2i\lambda a) \equiv \frac{\lambda a}{\operatorname{sh} \pi\tau} e^{i\lambda a} \Phi(1 + i\tau, 2; -2i\lambda a)$$

The last identity follows from the Kummer transformation /2, 3/ for the degenerate hypergeometric function.

The boundary value problems for a cresent domain containing an infinitely remote point are solved conveniently in bipolar coordinates α, σ

$$x = \frac{a \operatorname{sh} \alpha}{\operatorname{ch} \alpha - \cos \sigma}, \quad y = \frac{a \sin \sigma}{\operatorname{ch} \alpha - \cos \sigma}$$

$$(-\infty < \alpha < \infty, -\pi \leqslant \sigma \leqslant \pi, a > 0)$$

The quantity σ is measured by the angle between the tangent to the arc at the point $x = a$, $y = 0$ and the ray (a, ∞) on the x axis corresponding to the value $\sigma = 0$.

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Theorem 2. For $-\infty < y < \infty$ the following equations hold

$$\operatorname{sh}\lambda\sigma\begin{Vmatrix}\cos\lambda\alpha\\ \sin\lambda\alpha\end{Vmatrix}=\operatorname{sgn} y\int_{-\infty}^{\infty}A(\lambda,s)e^{-|sy|}\begin{Vmatrix}\cos xs\\ \sin xs\end{Vmatrix}ds \tag{1.3}$$

$$\operatorname{ch}\lambda\sigma\begin{Vmatrix}\cos\lambda\alpha\\ \sin\lambda\alpha\end{Vmatrix}-\begin{Vmatrix}1\\ 0\end{Vmatrix}=\int_{-\infty}^{\infty}B(\lambda,s)e^{-|sy|}\begin{Vmatrix}\cos xs\\ \sin xs\end{Vmatrix}ds$$

$$A(\lambda, s) = a\lambda e^{-isa}\Phi(1-i\lambda, 2;\ 2isa),\quad B(\lambda, s) = \operatorname{sgn} s A(\lambda, s)$$

Formulas (1.2) and (1.3) have been established by solving special boundary value problems for the Laplace equation.

Let us present the derivation of the last formula from (1.2). To this end, we consider the following internal Dirichlet problem for a symmetric cresent G bounded by arcs of the circles $\beta=\beta_0$ and $\beta=-\beta_0$.

$$\Delta w(\alpha,\beta)=0,\quad w(\alpha,\pm\beta_0)=\operatorname{ch}\lambda y\sin\lambda x - xa^{-1}\sin\lambda a \tag{1.4}$$

$$\left(x=\frac{a\operatorname{sh}\alpha}{\operatorname{ch}\alpha+\cos\beta_0},\ y=\frac{a\sin\beta_0}{\operatorname{ch}\alpha+\cos\beta_0},\ 0<\beta_0<\pi\right)$$

Evidently, $y\to\theta$, $x=\pm a$, $\cosh\lambda y\sin\lambda x - xa^{-1}\sin\lambda a\to 0$ as $\alpha\to\pm\infty$. The solution of problem (1.4) exists and is unique in the class $C^2(G)\cap C(\overline{G})$ and can be represented in the form

$$w(\alpha,\beta)=\int_0^\infty R(\lambda,\tau)\operatorname{ch}\tau\beta\sin\tau\alpha\,d\tau$$

$$R(\lambda,\tau)=\frac{2}{\pi}\frac{1}{\operatorname{ch}\tau\beta_0}\int_0^\infty p(\lambda,\alpha)\sin\tau\alpha\,d\alpha$$

$$p(\lambda,\alpha)=\operatorname{ch}\lambda y\sin\lambda x - xa^{-1}\sin\lambda a$$

(the functions $x=x(\alpha,\beta_0)$, $y=y(\alpha,\beta_0)$ are presented in (1.4)).

On the other hand, the function

$$\operatorname{ch}\lambda y\sin\lambda x - xa^{-1}\sin\lambda a\in C^2(G)\cap C(\overline{G})$$

is a solution of problem (1.4) (x and y are defined by the relationships (1.1)). Because of the uniqueness of the solution of problem (1.4) in the class $C^2(G)\cap C(\overline{G})$

$$\int_0^\infty R(\lambda,\tau)\operatorname{ch}\tau\beta\sin\tau\alpha\,d\tau=\operatorname{ch}\lambda y\sin\lambda x - xa^{-1}\sin\lambda a \tag{1.5}$$

everywhere in the domain G.

It is possible to start from (1.5) in any set $E\subset G$ having at least one finite limit point $P\in G$ when actually seeking the functions $R(\lambda,\tau)$. Setting $\beta=0$, $-\infty<\alpha<\infty$ ($y=0$, $-a<x<a$), applying the Fourier sine-transform inversion formula, and integrating by parts, we find after elementary reduction

$$R(\lambda,\tau)=\frac{\lambda}{2\pi\tau}\left[\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{i\tau}e^{-i\lambda x}dx+\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{i\tau}e^{i\lambda x}dx\right]-\frac{2\sin\lambda\alpha}{\operatorname{sh}\pi\tau}$$

Making the substitution $x=ay$ here, using the equation /3/

$$\int_{-1}^{1}(1-y)^{\nu-1}(1+y)^{\mu-1}e^{-ipy}dy=2^{\nu+\mu-1}B(\nu,\mu)e^{ip}\Phi(\mu,\nu+\mu;-2ip)$$

and the Kummer transformation, we have

$$R(\lambda,\tau)=\frac{\lambda a}{\operatorname{sh}\pi\tau}e^{-i\lambda a}\left[\Phi(1-i\tau,2;2i\lambda a)+\Phi(1+i\tau,2;2i\lambda a)\right]-\frac{2\sin\lambda a}{\operatorname{sh}\pi\tau}$$

Taking into account that the constructions carried out are valid for any value of β_0 in the interval $(0,\pi)$ and

$$2\sin\lambda a\int_0^\infty\frac{\operatorname{ch}\tau\beta\sin\tau\alpha}{\operatorname{sh}\pi\tau}d\tau=\frac{\operatorname{sh}\alpha\sin\lambda a}{\operatorname{ch}\alpha+\cos\beta}=\frac{x}{a}\sin\lambda a\quad(|\beta|<\pi)$$

we obtain the desired equality.

Representation of the function $e^{-i\mu a}\Phi(1-ip,2;2i\mu a)$ in terms of the regular Coulomb wave function /4/ enables us to write it in the form of the following series:

$$e^{-i\mu a}\Phi(1-ip,2;2i\mu a)=\sum_{n=1}^{\infty}A_n(p)(\mu a)^{n-1}$$

$$A_1(p)=1,\quad A_2(p)=p,\quad A_n(p)=\frac{2pA_{n-1}(p)-A_{n-2}(p)}{n(n-1)}\ (n>2)$$

It hence follows that the densities $C(\lambda, \tau)$ and $A(\lambda, s)$ are real functions for real values of λ, s, τ.

Formulas (1.2) and (1.3) and their specific combinations are specially adapted to the solution of boundary value problems of potential theory in the strip $-b \leqslant y \leqslant h$ (the half-plane $-b \leqslant y < \infty$) $(h > 0, b > 0)$ with a crescent hole or inclusion $\sigma_1 < \sigma < \sigma_2$, and in particular, to investigating singularities of the fields being studied near the angular points $x = \pm a,\ y = 0$.

The following can be considered as initial results when solving the problems mentioned in the strip $-b \leqslant x \leqslant h$ (the half-plane $-b \leqslant h < \infty$) with the previous orientation of the crescent hole or inclusion.

Theorem 3. For $-\pi < \beta < \pi$ the following equations hold

$$\left\|\begin{matrix}\operatorname{ch}\lambda x\cos\lambda y\\ \operatorname{sh}\lambda x\sin\lambda y\end{matrix}\right\| - \left\|\begin{matrix}\operatorname{ch}\lambda a\\ 0\end{matrix}\right\| = \pm i\int_{-\infty}^{\infty} G(\lambda,\tau)\left\|\begin{matrix}\operatorname{ch}\tau\beta\cos\tau\alpha\\ \operatorname{sh}\tau\beta\sin\tau\alpha\end{matrix}\right\| d\tau$$

$$\left\|\begin{matrix}\operatorname{ch}\lambda x\sin\lambda y\\ \operatorname{sh}\lambda x\cos\lambda y\end{matrix}\right\| = \int_{-\infty}^{\infty} G(\lambda,\tau)\left\|\begin{matrix}\operatorname{sh}\tau\beta\cos\tau\alpha\\ \operatorname{ch}\tau\beta\sin\tau\alpha\end{matrix}\right\| d\tau$$

$$G(\lambda,\tau) = \frac{\lambda a}{\operatorname{sh}\pi\tau} e^{\lambda a}\Phi(1 - i\tau, 2; -2\lambda a)$$

Theorem 4. For $|x| > a$ the following equations hold

$$\left\|\begin{matrix}\operatorname{ch}\lambda\sigma\\ \operatorname{sh}\lambda\sigma\end{matrix}\right\|\sin\lambda\alpha = \pm\operatorname{sgn}x\int_{-\infty}^{\infty} a(\lambda,s)e^{-|sx|}\left\|\begin{matrix}\cos sy\\ i\sin sy\end{matrix}\right\| ds$$

$$\left\|\begin{matrix}\operatorname{ch}\lambda\sigma\\ \operatorname{sh}\lambda\sigma\end{matrix}\right\|\cos\lambda\alpha - \left\|\begin{matrix}1\\ 0\end{matrix}\right\| = \int_{-\infty}^{\infty} b(\lambda,s)e^{-|sx|}\left\|\begin{matrix}i\cos sy\\ \sin sy\end{matrix}\right\| ds$$

$$a(\lambda, s) = a\lambda e^{sa}\Phi(1 - i\lambda, 2; -2sa),\quad b(\lambda, s) = \operatorname{sgn} sa(\lambda, s)$$

Note that the functions

$$G(\lambda,\tau) + G(\lambda,-\tau),\ i[G(\lambda,\tau) - G(\lambda,-\tau)] \qquad a(\lambda,s) + a(\lambda,-s),\ i[a(\lambda,s) - a(\lambda,-s)]$$

are real for real values of λ, s, and τ.

2. As the simplest example of the application of the equations obtained we consider the problem of antiplane deformation of a layer $(-\infty < x, z < \infty, -b \leqslant y \leqslant b)$ which is symmetric in the x coordinate and a crescent profile weakened by a cylindrical channel $(-\infty < z < \infty, -\infty < \alpha < \infty, -\sigma_0 < \sigma < \sigma_0)$. Layer deformation is caused by shearing loads applied on the faces $y = \pm b$ which are directed along and are constant on the lines $y = \pm b,\ x = \text{const}$. It is known /5, 6/ that in this case only the displacement $w = w(x, y)$ along the z axis and the tangential stresses (G is the shear modulus) can be considered to be different from zero

$$\tau_{xz} = G\partial w/\partial x,\ \tau_{yz} = G\partial w/\partial y$$

The system of equilibrium equations reduces to one equation which takes the form $\Delta w = 0$ in the absence of mass forces.

Let the surface of the cylindrical channel be free of external forces. We separate the problem into symmetric and antisymmetric problems in the y coordinate and consider the former when the desired function w is even in the y coordinate. Then determination of the stress and displacement fields in the body under consideration (taking into account the assumed symmetry of the problem in the x coordinate also) reduces to solving a Neumann problem in the plane domain D bounded by the lines $y = \pm b$ and the arcs of the intersecting circles $\sigma = \pm\sigma_0$ (a strip with a crescent hole)

$$\Delta w = 0,\quad \left.\frac{\partial w}{\partial \sigma}\right|_{\sigma=\pm\sigma_0} = 0,\quad \left.\frac{\partial w}{\partial y}\right|_{y=\pm b} = \pm f_1(x),\quad f_1(-x) = f_1(x) \tag{2.1}$$

It is assumed that the external forces applied to the layer boundary are equalized. This results in this case in the condition

$$\int_0^\infty f_1(x)\,dx = 0 \tag{2.2}$$

which is simultaneously also the necessary condition for the two-dimensional Neumann problem (2.1) to be solvable.

We seek the solution of the problem in the class of functions satisfying the condition of finiteness of the elastic strain energy /7, 8/ of the strip $D\ (-\infty < x < \infty, -b \leqslant y \leqslant b)$ weakened by a crescent hole $-\sigma_0 < \sigma < \sigma_0$. The energy mentioned is stored in the domain D because of the work of the external forces which is naturally always considered to be finite /7, 8/.

Therefore, the solution of problem (2.1) should be sought in the class of functions satisfying the condition

$$\iint\limits_{(D)} \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy < \infty \tag{2.3}$$

Condition (2.3) completely closes the formulation of problem (2.1), (2.2). Meanwhile, the behaviour of the solution is determined exactly as $|x| \to \infty$, as well as at the singularities $x = \pm a,\ y = 0$.

Without examining the formulation of general assumptions relative to the external loads, we note the following. If the function $f_1(x)$ is bounded and non-zero just in a finite interval, or $f_1(x)$ is integrable in the interval $(0, \infty)$ and $f_1(x) = O(x^{-1-\varepsilon})$ $(x \to \infty, \varepsilon \geqslant 1)$, then the constructions performed here are valid and condition (2.3) holds.

We represent the harmonic function w in the form

$$w = \int_0^\infty G_1(\lambda)(\operatorname{ch}\lambda y \cos\lambda x - \cos\lambda a)\,d\lambda + \int_0^\infty H_1(\tau)\operatorname{ch}\tau\sigma\cos\tau\alpha\,d\tau + \text{const}$$

Taking account of the symmetry of the problem and the equations

$$\beta = \pi - \sigma \quad (0 < \sigma \leqslant \pi)$$

$$\frac{\partial w}{\partial \sigma} = -\int_0^\infty G_1(\lambda)\,d\lambda\int_0^\infty \tau\,[C(\lambda,\tau) + C(\lambda,-\tau)]\operatorname{sh}\tau\beta\cos\tau\alpha\,d\tau + \int_0^\infty \tau H_1(\tau)\operatorname{sh}\tau\sigma\cos\tau\alpha\,d\tau$$

$$\frac{\partial w}{\partial y} = \int_0^\infty \lambda G_1(\lambda)\operatorname{sh}\lambda y\cos\lambda x\,d\lambda - \int_0^\infty H_1(\tau)\,d\tau\int_0^\infty \lambda\,[B(\tau,\lambda) + B(\tau,-\lambda)]\,e^{-\lambda y}\cos\lambda x\,d\lambda \quad (y > 0)$$

on satisfying the boundary conditions of the problem, we arrive at the relations

$$H_1(\tau) = \frac{\operatorname{sh}\tau(\pi - \sigma_0)}{\operatorname{sh}\tau\sigma_0}\int_0^\infty G_1(\lambda)\,[C(\lambda,\tau) + C(\lambda,-\tau)]\,d\lambda$$

$$G_1(\lambda) = \frac{e^{-\lambda b}}{\operatorname{sh}\lambda b}\int_0^\infty H_1(\tau)\,[B(\tau,\lambda) + B(\tau,-\lambda)]\,d\tau + \frac{p_1(\lambda)}{\lambda\operatorname{sh}\lambda b}$$

$$p_1(\lambda) = \frac{2}{\pi}\int_0^\infty f_1(x)\cos\lambda x\,dx, \quad p_1(0) = 0$$

Eliminating $H_1(\tau)$, setting $G_1(\lambda) = \lambda^{-1}\psi_1(\lambda)$ and applying the Kummer transformation, we obtain an integral equation after certain calculations ($M_{\lambda,\nu}(z)$ is a Whittaker function /2/)

$$\psi_1(\lambda) = \int_0^\infty K(\lambda, u)\,\psi_1(u)\,du + \frac{p_1(\lambda)}{\operatorname{sh}\lambda b} \quad (\lambda > 0) \tag{2.4}$$

$$K(\lambda, u) = -\frac{e^{-\lambda b}}{4u\operatorname{sh}\lambda b}\int_{-\infty}^\infty \frac{\tau\operatorname{sh}\tau(\pi - \sigma_0)}{\operatorname{sh}\tau\sigma_0\operatorname{sh}\pi\tau}W(\lambda, u; \tau)\,d\tau$$

$$W(\lambda, u; \tau) = [M_{i\tau,1/2}(2iua) + M_{i\tau,1/2}(-2iua)]\,M_{i\tau,1/2}(2i\lambda a)$$

To extract the principal part of the kernel $K(\lambda, u)$ as $\lambda + u \to \infty$ and $0 < \sigma_0 \leqslant \pi/2$ we apply the equation

$$\frac{\operatorname{sh}\tau(\pi - \sigma_0)}{\operatorname{sh}\tau\sigma_0} = 2\operatorname{sh}[|\tau|(\pi - \sigma_0)]\sum_{n=0}^{N} e^{-(2n+1)\sigma_0|\tau|} + \frac{\operatorname{sh}\tau(\pi - \sigma_0)}{\operatorname{sh}\tau\sigma_0}e^{-(2N+2)\sigma_0|\tau|} \tag{2.5}$$

the representation /9/

$$M_{i\tau,1/2}(\alpha)\,M_{i\tau,1/2}(\beta) = \frac{\operatorname{sh}\pi\tau}{\pi\tau}\sqrt{\alpha\beta}\int_{-\infty}^\infty e^{2i\rho\tau}e^{-1/2(\alpha+\beta)\operatorname{th}\rho}J_1\left(\frac{\sqrt{\alpha\beta}}{\operatorname{ch}\rho}\right)\frac{d\rho}{\operatorname{ch}\rho} \tag{2.6}$$

and its resulting relationship

$$\int_{-\infty}^\infty e^{-2i\rho\tau}\frac{\tau}{\operatorname{sh}\pi\tau}M_{i\tau,1/2}(\alpha)\,M_{i\tau,1/2}(\beta)\,d\tau = \frac{\sqrt{\alpha\beta}}{\operatorname{ch}\rho}e^{-1/2(\alpha+\beta)\operatorname{th}\rho}J_1\left(\frac{\sqrt{\alpha\beta}}{\operatorname{ch}\rho}\right) \tag{2.7}$$

Taking account of the structure of the function $W(\lambda, u; \tau)$ and the inequality $|J_1(x)| < 1$ we obtain by using the representation (2.6)

$$|W(\lambda, u; \pm\tau)| \leqslant 2a\sqrt{\lambda u}\,\tau^{-1}\operatorname{sh}\pi\tau\,[I_1(2a\sqrt{\lambda u}) + 1] \quad (\lambda, u \geqslant 0) \tag{2.8}$$

It can be established that for any fixed σ_0 $(0 < \sigma_0 \leqslant \pi/2)$ the number $N = N(\sigma_0)$ and the number $\omega = \omega(\sigma_0)$ $(0 \leqslant \omega < 2\sigma_0)$ are determined uniquely such that $(2N + 2)\sigma_0 = \pi - \omega$. The solution satisfying these conditions has the form

$$N = \left[\frac{\pi}{2\sigma_0}\right] - 1, \quad \omega = \pi - 2\left[\frac{\pi}{2\sigma_0}\right]\sigma_0$$

Considering the number N in (2.5) to have been selected in precisely such a manner, we introduce the following notation

$$V(\lambda, u) = \int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} \frac{\operatorname{sh}\tau(\pi-\sigma_0)}{\operatorname{sh}\tau\sigma_0} W(\lambda, u; \tau)\, d\tau = V_N^{(1)}(\lambda, u) + V_N^{(2)}(\lambda, u)$$

$$V_N^{(1)}(\lambda, u) = \int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} \frac{\operatorname{sh}\tau(\pi-\sigma_0)}{\operatorname{sh}\tau\sigma_0} e^{-(2N+2)\sigma_0|\tau|} W(\lambda, u; \tau)\, d\tau$$

$$V_N^{(2)}(\lambda, u) = 2\sum_{n=0}^{N} \int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} \operatorname{sh}[|\tau|(\pi-\sigma_0)]\, e^{-(2n+1)\sigma_0|\tau|} W(\lambda, u; \tau)\, d\tau$$

In conformity with the inequality (2.8), we have ($\psi(x)$ is Euler's psi-function /3/):

$$|V_N^{(1)}(\lambda, u)| \leqslant \frac{a}{\sigma_0}\left[\psi\left(N+1+\frac{\pi}{2\sigma_0}\right) - \psi\left(N+2-\frac{\pi}{2\sigma_0}\right)\right]\sqrt{\lambda u}\,[I_1(2a\sqrt{\lambda u})+1]$$

We write the function $V_N^{(2)}(\lambda, u)$ in the form

$$V_N^{(2)}(\lambda, u) = \sum_{n=0}^{N}\sum_{m=1}^{3} T_n^{(m)}(\lambda, u)$$

$$T_n^{(1)}(\lambda, u) = \int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} e^{[\pi-(2n+2)\sigma_0]\tau}[W(\lambda, u; \tau) + W(\lambda, u; -\tau)]\, d\tau$$

$$T_n^{(2)}(\lambda, u) = -\int_{0}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} e^{-[\pi-(2n+2)\sigma_0]\tau}[W(\lambda, u; \tau) + W(\lambda, u; -\tau)]\, d\tau$$

$$T_n^{(3)}(\lambda, u) = -\int_{0}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} e^{-(\pi+2n\sigma_0)\tau}[W(\lambda, u; \tau) + W(\lambda, u; -\tau)]\, d\tau$$

Taking account of the estimate (2.8), we have

$$|T_n^{(3)}(\lambda, u)| \leqslant \frac{4a\sqrt{\lambda u}}{\pi + 2n\sigma_0}[I_1(2a\sqrt{\lambda u}) + 1] \quad (0 \leqslant n \leqslant N)$$

Let $\omega > 0$. In this case

$$\pi - (2n+2)\sigma_0 \geqslant \omega > 0 \tag{2.9}$$

$$|T_n^{(2)}(\lambda, u)| \leqslant \frac{4a\sqrt{\lambda u}}{\pi - (2n+2)\sigma_0}[I_1(2a\sqrt{\lambda u}) + 1] \quad (0 \leqslant n \leqslant N)$$

If $\omega = 0$, then $(2N+2)\sigma_0 = \pi$, $(2n+2)\sigma_0 < \pi$, $\pi - (2n+2)\sigma_0 > 0$ and therefore, the estimate (2.9) holds for $0 \leqslant n \leqslant N-1$ (if $N \geqslant 1$). For $\omega = 0$ and $n = N$, we have

$$(2N+2)\sigma_0 = \pi, \; T_N^{(2)}(\lambda, u) = 2a\sqrt{\lambda u}\,[I_1(2a\sqrt{\lambda u}) - J_1(2a\sqrt{\lambda u})]$$

Now utilizing (2.7), we obtain the following representation for the quantities $T_n^{(1)}(\lambda, u)$

$$T_n^{(1)}(\lambda, u) = P_n(\lambda, u) + P_n(\lambda, -u) + P_n(-\lambda, u) + P_n(-\lambda, -u)$$

$$P_n(\lambda, u) = -\frac{2a\sqrt{\lambda u}}{\sin(n+1)\sigma_0}\exp[a(\lambda+u)\operatorname{ctg}(n+1)\sigma_0]\, I_1\left(\frac{2a\sqrt{\lambda u}}{\sin(n+1)\sigma_0}\right)$$

Since $(2N+2)\sigma_0 = \pi - \omega\ (0 \leqslant \omega < 2\sigma_0)$, then

$$0 < (n+1)\sigma_0 \leqslant \pi/2\ (0 \leqslant n \leqslant N), \; \operatorname{ctg}(n+1)\sigma_0 \geqslant 0$$

$$T_n^{(1)}(\lambda, u) \sim P_n(\lambda, u)\ (\lambda + u \to \infty,\ (n+1)\sigma_0 < \pi/2)$$

and

$$T_n^{(1)}(\lambda, u) \sim -4a\sqrt{\lambda u}\, I_1(2a\sqrt{\lambda u}) \quad (\lambda + u \to \infty)$$

if $(n+1)\sigma_0 = \pi/2$, i.e. $\omega = 0$ for $n = N$.

The estimates obtained show that when $0 < \sigma_0 < \pi/2$

$$V(\lambda, u) \sim P_0(\lambda, u)\ (\lambda + u \to \infty)$$

For the value $\sigma_0 = \pi/2$ the function $V(\lambda, u)$ is found exactly

$$V(\lambda, u) = -2a\sqrt{\lambda u}\,[I_1(2a\sqrt{\lambda u}) - J_1(2a\sqrt{\lambda u})]$$

Therefore, for $0 < \sigma_0 \leqslant \pi/2$

$$K(\lambda, u) \sim \frac{a}{2\sin\sigma_0}\frac{e^{-\lambda b}}{\operatorname{sh}\lambda b}\sqrt{\frac{\lambda}{u}}\, e^{a(\lambda+u)\operatorname{ctg}\sigma_0} I_1\left(\frac{2a\sqrt{\lambda u}}{\sin\sigma_0}\right) \quad (\lambda + u \to \infty)$$

$$K(\lambda, \lambda) \sim \frac{1}{4}\sqrt{\frac{a}{\pi\lambda\sin\sigma_0}}\frac{e^{-\lambda b}}{\operatorname{sh}\lambda b}\, e^{2\lambda a\operatorname{ctg}{}^1/{}_2\sigma_0} \quad (\lambda \to \infty)$$

For the values $\pi/2 < \sigma_0 < \pi$ the method presented for obtaining the asymptotic form of the kernel $K(\lambda, u)$ is evidently inapplicable. We obtain the estimate of $K(\lambda, u)$ corresponding to this case as $\lambda + u \to \infty$ by starting from the equation

$$\frac{\operatorname{sh}\tau(\pi-\sigma_0)}{\operatorname{sh}\tau\sigma_0} = e^{(\pi-2\sigma_0)|\tau|} - e^{-\pi|\tau|} + \frac{\operatorname{sh}\tau(\pi-\sigma_0)}{\operatorname{sh}\tau\sigma_0} e^{-2\sigma_0|\tau|}$$

Itilizing the inequalities

$$\frac{\operatorname{sh}\pi\tau}{\pi\tau} \leqslant \operatorname{ch}\pi\tau, \quad e^{-2\sigma_0\tau}\operatorname{ch}\pi\tau \leqslant 1 \quad \left(\tau \geqslant 0,\ \sigma_0 > \frac{\pi}{2}\right)$$

and the estimate

$$|M_{\pm i\tau, 1/2}(is)| \leqslant s\,\frac{\operatorname{sh}\pi\tau}{\pi\tau}$$

resulting from the integral representation of the Whittaker function /3/

$$M_{\pm i\tau, 1/2}(is) = \frac{1}{2} is \frac{\operatorname{sh}\pi\tau}{\pi\tau} \int_{-1}^{1} (1+t)^{\mp i\tau}(1-t)^{\pm i\tau} e^{1/2 ist}\, dt$$

we have the following inequalities:

$$|W(\lambda, u; \pm\tau)| \leqslant 8a^2\lambda u \left(\frac{\operatorname{sh}\pi\tau}{\pi\tau}\right)^2 \quad (\lambda, u \geqslant 0) \tag{2.10}$$

$$\left|\int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau}\, \frac{\operatorname{sh}\tau(\pi-\sigma_0)}{\operatorname{sh}\tau\sigma_0} e^{-2\sigma_0|\tau|} W(\lambda, u; \tau)\, d\tau\right| \leqslant \frac{16a^2}{\sigma_0} \lambda u \operatorname{tg}\frac{\pi(\pi-\sigma_0)}{2\sigma_0}$$

Furthermore, by using inequalities (2.8), (2.10) and sinh $\pi\tau \leqslant \pi\tau$ cosh $\pi\tau$ we obtain

$$\left|\int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} e^{-\pi|\tau|} W(\lambda, u; \tau)\, d\tau\right| \leqslant \frac{8a^2\lambda u}{\pi} \int_0^{a\sqrt{\lambda u}} \frac{\operatorname{sh}\pi\tau}{\pi\tau} e^{-\pi\tau}\, d\tau +$$

$$4a\sqrt{\lambda u}\,[I_1(2a\sqrt{\lambda u}) + 1] \int_{a\sqrt{\lambda u}}^{\infty} e^{-\pi\tau}\, d\tau \leqslant \frac{8a^3}{\pi}(\lambda u)^{3/2} + \frac{4a}{\pi}\sqrt{\lambda u}\,[I_1(2a\sqrt{\lambda u}) + 1]\, e^{-\pi a\sqrt{\lambda u}}$$

We now examine the integral

$$S(\lambda, u) = \int_{-\infty}^{\infty} \frac{\tau}{\operatorname{sh}\pi\tau} e^{(\pi-2\sigma_0)|\tau|} W(\lambda, u; \tau)\, d\tau \quad (\lambda, u \geqslant 0)$$

Setting $\sigma_0 = \pi/2 + \varepsilon_0$ $(0 < \varepsilon_0 < \pi/2)$ therein, taking account of the structure of the function $W(\lambda, u; \tau)$, Eq. (2.6), and integrating with respect to τ, to find

$$S(\lambda, u) = S_1(\lambda, u) + S_1(\lambda, -u)$$

$$S_1(\lambda, u) = -\frac{2a\varepsilon_0}{\pi}\sqrt{\lambda u} \int_{-\infty}^{\infty} e^{-ia(\lambda+u)\operatorname{th}\rho}\, I_1\left(\frac{2a\sqrt{\lambda u}}{\operatorname{ch}\rho}\right) \frac{d\rho}{(\varepsilon_0^2+\rho^2)\operatorname{ch}\rho}$$

Using the inequalities $|J_1(x)| < 1$, $\varepsilon_0^2 + \rho^2 \geqslant \varepsilon_0^2$, we have the following estimate

$$|S_1(\lambda; -u)| \leqslant 2a\varepsilon_0^{-1}\sqrt{\lambda u}$$

To estimate $S_1(\lambda, u)$ we apply the method of contour integration. To this end we set $z = \rho + i\delta$ and we examine the domain $\Omega = \Omega_1 \setminus \Omega_2$, where Ω_1 is rectangular, $(-R < \rho < R, 0 < \delta < \pi/2)$, Ω_2 is the semicircle $(\rho^2 + \delta^2 < \gamma^2)$, and $\delta > 0$, $\gamma < \pi/2 - \varepsilon_0$, $\gamma < R$, $\gamma > 0$.

We introduce the function

$$f(z) = -e^{-ia(\lambda+u)\operatorname{cth} z}\, J_1\left(\frac{2a\sqrt{\lambda u}}{\operatorname{sh} z}\right) \frac{1}{[(z-i\pi/2)^2 + \varepsilon_0^2]\operatorname{sh} z}$$

which is analytic in the domain Ω and on the boundary Γ except the point $z = i(\pi/2 - \varepsilon_0) \in \Omega$. Applying the residue theorem to the integral of the function $f(z)$ along the contour Γ and passing to the limit as $\gamma \to 0$, $R \to \infty$, we find

$$S_1(\lambda, u) = -\frac{2a}{\cos\varepsilon_0}\sqrt{\lambda u}\, e^{-a(\lambda+u)\operatorname{tg}\varepsilon_0} I_1\left(\frac{2a\sqrt{\lambda u}}{\cos\varepsilon_0}\right) + O(\sqrt{\lambda u})$$

Therefore, for $\pi/2 < \sigma_0 < \pi$

$$K(\lambda, u) = \sqrt{\frac{\lambda}{u}}\, \frac{e^{-\lambda b}}{\operatorname{sh}\lambda b} \left[\frac{a}{2\sin\sigma_0} e^{a(\lambda+u)\operatorname{ctg}\sigma_0} I_1\left(\frac{2a\sqrt{\lambda u}}{\sin\sigma_0}\right) + O(\lambda u)\right]$$

$$(\lambda - u \to \infty)$$

$$K(\lambda, \lambda) \sim \frac{1}{4}\sqrt{\frac{a}{\pi\lambda\sin\sigma_0}}\, \frac{e^{-\lambda b}}{\operatorname{sh}\lambda b}\, e^{2\lambda a\operatorname{ctg}{}^1\!/_2\sigma_0} \quad (\lambda \to \infty)$$

The estimates obtained for the kernel $K(\lambda, u)$ as $\lambda + u \to \infty$ show that (2.4) reduces to a Fredholm equation (the kernel and free term are square-summable) by replacing the desired function $\psi_1(\lambda)$ only in the case when $b > a \operatorname{ctg} {}^1/_2\sigma_0$.

Geometrically, this condition means that the crescent contour should lie entirely in the strip $-b < y < b$.

Transformation of (2.4) to a Fredholm equation is achieved by the substitution, for example

$$\psi_1(\lambda) = e^{-\lambda a \operatorname{ctg} {}^1/_2\sigma_0} \varphi_1(\lambda)$$

In certain special cases the kernel of the integral equation (2.4) is calculated in closed form. For $\sigma_0 = \pi/3$ it has the form

$$K(\lambda, u) = \frac{2a}{\sqrt{3}} \frac{e^{-\lambda b}}{\operatorname{sh} \lambda b} \sqrt{\frac{\lambda}{u}} \left[\operatorname{ch} \frac{a(\lambda + u)}{\sqrt{3}} I_1\left(\frac{4a\sqrt{\lambda u}}{\sqrt{3}}\right) - \operatorname{ch} \frac{a(\lambda - u)}{\sqrt{3}} I_1\left(\frac{4a\sqrt{\lambda u}}{\sqrt{3}}\right)\right] \geqslant 0$$

while for $\sigma_0 = \pi/2$

$$K(\lambda, u) = \frac{a}{2} \sqrt{\frac{\lambda}{u}} \frac{e^{-\lambda b}}{\operatorname{sh} \lambda b} [I_1(2a\sqrt{\lambda u}) - J_1(2a\sqrt{\lambda u})] \geqslant 0$$

We explain the behaviour of the solution of the problem as $\rho \to \infty$ $(\rho = \sqrt{x^2 + y^2})$ by considering, say, that

$$f_1(x) = \frac{\varphi(x)}{1 + x^{1+\varepsilon}} \ (1 \leqslant \varepsilon < 2), \quad |\varphi(x)| \leqslant M, \quad \lim_{x \to \infty} \varphi(x) = C < \infty$$

In this case, from the condition

$$\int_0^\infty f_1(x)\, dx = 0$$

it follows that

$$\lim_{\lambda \to 0} \frac{p_1(\lambda)}{\lambda^\varepsilon} = -\frac{2}{\pi} C \int_0^\infty \frac{1 - \cos u}{u^{1+\varepsilon}} du \quad (1 \leqslant \varepsilon < 2)$$

Therefore, $p_1(\lambda) = O(\lambda^\varepsilon)$ $(\lambda \to 0, 1 \leqslant \varepsilon < 2)$ and we have for $\varepsilon = 1$

$$G_1(\lambda) = C_1\lambda^{-1} + g_1(\lambda), \quad g_1(\lambda) = o(\lambda^{-1}) \ (\lambda \to 0,\ C_1 = \text{const})$$

$$\int_0^\infty G_1(\lambda)(\operatorname{ch} \lambda y \cos \lambda x - \cos \lambda a)\, d\lambda = \int_0^\infty G_1(\lambda)(\cos \lambda x - \cos \lambda a)\, d\lambda +$$

$$\int_0^\infty G_1(\lambda)(\operatorname{ch} \lambda y - 1) \cos \lambda x\, d\lambda = C_1 \ln \frac{a}{|x|} + O(1) \quad (\rho \to \infty)$$

$$w = C_1 \ln \frac{a}{|x|} + O(1), \quad \frac{\partial w}{\partial x} = O\left(\frac{1}{x}\right), \quad \frac{\partial w}{\partial y} = o\left(\frac{1}{x}\right) \quad (\rho \to \infty)$$

For $\varepsilon > 1$ the singularity of the function $G_1(\lambda)$ at zero is integrable, and then

$$\lim_{\rho \to \infty} \int_0^\infty G_1(\lambda)(\operatorname{ch} \lambda y \cos \lambda x - \cos \lambda a)\, d\lambda = -\int_0^\infty G_1(\lambda) \cos \lambda a\, d\lambda$$

$$w = O(1), \quad \frac{\partial w}{\partial x} = o\left(\frac{1}{x}\right), \quad \frac{\partial w}{\partial y} = o\left(\frac{1}{x}\right) \quad (\rho \to \infty)$$

We now examine the antisymmetric problem when the desired function w is odd in the y coordinate and

$$\left.\frac{\partial w}{\partial y}\right|_{y=\pm b} = f_2(x), \quad f_2(-x) = f_2(x)$$

In this case the condition of statics is satisfied identically.

Representing the harmonic function w in the form

$$w = \int_0^\infty G_2(\lambda) \operatorname{sh} \lambda y \cos \lambda x\, d\lambda + \int_0^\infty H_2(\tau) \operatorname{sh} \tau\beta \cos \tau\alpha\, d\tau$$

and satisfying the boundary conditions of the problem, we obtain the relationships

$$G_2(\lambda) = \frac{e^{-\lambda b}}{\operatorname{ch} \lambda b} \int_0^\infty H_2(\tau)[A(\tau, \lambda) + A(\tau, -\lambda)]\, d\tau + \frac{p_2(\lambda)}{\lambda \operatorname{ch} \lambda b}$$

$$H_2(\tau) = \frac{\operatorname{ch} \tau(\pi - \sigma_0)}{\operatorname{ch} \tau\sigma_0} \int_0^\infty G_2(\lambda)[C(\lambda, \tau) - C(\lambda, -\tau)]\, d\lambda, \qquad p_2(\lambda) = \frac{2}{\pi} \int_0^\infty f_2(x) \cos \lambda x\, dx$$

The question of obtaining a Fredholm integral equation of the second kind in the function $G_2(\lambda)$ and the necessity of the condition $b > a\,\mathrm{ctg}\,{}^1/_2\sigma_0$ in this connection is solved in exactly the same way as in the case of the symmetric problem.

We also note that if the function $f_2(x)$ is absolutely integrable, then

$$\frac{\partial w}{\partial x} = o\left(\frac{1}{x}\right), \quad \frac{\partial w}{\partial y} = o\left(\frac{1}{x}\right) \quad (\rho \to \infty)$$

We investigate the behaviour of the shear stresses $\tau_{yz} = G\partial w/\partial y$ as we approach the angular points of the crescent $y = 0, x = \pm a$ $(\alpha \to \pm\infty)$. Taking into account the equation

$$\frac{\partial \alpha}{\partial y} = 0, \quad \frac{\partial \sigma}{\partial y} = \frac{\mathrm{ch}\,\alpha - 1}{a} \quad (\sigma = 0\,(y = 0;\ |x| > a))$$

we find

$$\left.\frac{\partial w}{\partial y}\right|_{\substack{y=0,\ |x|>a \\ (\sigma=0)}} = \int_0^\infty \lambda G_2(\lambda)\cos\lambda x\,d\lambda + \frac{\mathrm{ch}\,\alpha - 1}{a}\int_0^\infty \tau H_2(\tau)\cos\tau\alpha\,d\tau$$

Now utilizing the expression for $H_2(\tau)$ and the evenness of the function $\tau[C(\lambda, \tau) - C(\lambda, -\tau)]$ in τ we obtain after certain manipulations

$$\left.\frac{\partial w}{\partial y}\right|_{\substack{y=0,\ |x|>a \\ (\sigma=0)}} = \int_0^\infty \lambda G_2(\lambda)\cos\lambda x\,d\lambda + \frac{\mathrm{ch}\,\alpha - 1}{2}\int_0^\infty \lambda e^{i\lambda a}[R_\lambda^+(\alpha) + R_\lambda^-(\alpha)]\,G_2(\lambda)\,d\lambda$$

$$R_\lambda^\pm(\alpha) = \int_{-\infty}^{\infty} \frac{\tau\,\mathrm{ch}\,\tau(\pi - \sigma_0)}{\mathrm{sh}\,\pi\tau\,\mathrm{ch}\,\tau\sigma_0}\,\Phi(1 \pm i\tau, 2; -2i\lambda a)\,e^{i\tau\alpha}\,d\tau$$

Taking into account the asymptotic behaviour of the function $\Phi(c, \gamma; z)$ which is entire in the parameter c, as $c \to \infty$, by applying the residue theorem we have

$$R_\lambda^\pm(\alpha) = \left(\frac{\pi}{\sigma_0}\right)^2 \sum_{n=1}^{\infty}(2n-1)\,\Phi\left(1 \mp \frac{\pi(2n-1)}{2\sigma_0}, 2; -2i\lambda a\right)\times \exp\left[-\frac{\pi\alpha(2n-1)}{2\sigma_0}\right] - 2\sum_{n=1}^{\infty} n\Phi(1 \mp n, 2; -2i\lambda a)\,e^{-n\alpha}$$

Extracting the principal part $R_\lambda^\pm(\alpha)$ as $\alpha \to \infty$ (corresponding to the value $n = 1$), and using the equation

$$\Phi(2, 2; -2i\lambda a) = e^{-2i\lambda a}, \quad \Phi(0, 2; -2i\lambda a) = 1$$

we arrive at the following deductions.

For $\sigma_0 < \pi/2$, the stresses τ_{yz} at the angular points $x = \pm a, \mp y = 0$ of the domain D are zero. When $\sigma_0 = \pi/2$ the stresses τ_{yz} are bounded and

$$\lim_{x \to \pm a} \tau_{yz}\Big|_{\substack{\sigma=0 \\ (y=0)}} = 2G\int_0^\infty \lambda G_2(\lambda)\cos\lambda a\,d\lambda$$

For $\sigma_0 > \pi/2$ the stresses τ_{yz} at the angular points of the domain D increase without limit in absolute value, where

$$\tau_{yz}\Big|_{\substack{\sigma=0 \\ (y=0)}} \sim c\,(x - a)^{-1+{}^1/_2\pi/\sigma_0} \quad (x \to a,\ c = \mathrm{const})$$

The stress singularity clarified at the angular points of the domain D is maximal and its order agrees exactly with the order of the singularity in the problem of the longitudinal shear of a wedge $-\sigma_0 < \sigma < \sigma_0, -\infty < z < \infty, 0 < r < \infty$ and problems on the torsion and bending of rods with a section in the shape of a symmetric crescent /1, 6/.

By analogous constructions it can be seen that in the problem symmetric in the y coordinate considered above, the stresses at the angles of the crescent are zero for $0 < \sigma_0 \leqslant \pi$.

It follows from symmetry considerations that the results obtained simultaneously yield solutions of the first fundamental and mixed problems of antiplane strain of a strip $0 \leqslant y \leqslant b$ with a segmental recess. This explains the absence of stress singularities at angular points of the domain D in the symmetric problem and their presence in the antisymmetric problem. In fact, the stresses in the first fundamental problem of antiplane strain in the neighbourhood of an angular point with aperture angle $\sigma_0 \leqslant \pi$ are bounded /6, 8, 10/. In the mixed problem the aperture angle $\sigma_0 = \pi/2$ delimits the angles for which the stresses tend to zero as one approaches the angular point $(\sigma_0 < \pi/2)$ from the angles for which the stresses increase without limit $(\sigma_0 > \pi/2)$ /1, 6, 8, 10/.

The scheme described for solving the antiplane problem symmetric in the x coordinate is easily extended even to the case when a strip $-b \leqslant y \leqslant h$ with a crescent hole $-\sigma_1 < \sigma < \sigma_2$ is considered instead on the domain D. The harmonic function w can be selected in the form

$$w = \int_0^\infty G_1(\lambda)(\operatorname{ch}\lambda y\cos\lambda x - \cos\lambda a)\,d\lambda + \int_0^\infty G_2(\lambda)\operatorname{sh}\lambda y\cos\lambda x\,d\lambda + \int_0^\infty H_1(\tau)\operatorname{ch}\tau\sigma\cos\tau\alpha\,d\tau + \int_0^\infty H_2(\tau)\operatorname{sh}\tau\sigma\cos\tau\alpha\,d\tau + \text{const} \tag{2.11}$$

and instead of one integral equation (2.4) a system of two integral equations of the second kind in the functions $\psi_i(\lambda) = \lambda G_i(\lambda)$ $(i = 1, 2)$ is obtained. Investigation of the behaviour of the kernels of these equations for $\lambda + u \to \infty$ is analogous to that presented above.

Everything relative to the problems that are symmetric in the x coordinate is carried over completely to problems antisymmetric in the x coordinate. In this case, $\cos\lambda x$ in (2.11) must be replaced by $\sin\lambda x$ and $\cos\tau\alpha$ by $\sin\tau\alpha$.

Superposition of the solutions of the problems mentioned also enables us to consider problems in which the given loads are functions of a general kind. Moreover, the equations presented in Sect.1 enable the Dirichlet problem and the fundamental mixed problems of antiplane strain to be investigated for the domains mentioned, which in combination with the method of dual integral equations also enable intrinsically mixed (contact) problems to be studied. We note that in a number of cases the need arises to insert a logarithmic term of the form

$$B_0 \ln\frac{a^2}{x^2+y^2} \quad (B_0 = \text{const})$$

into the general solution.

REFERENCES

1. UFLIAND YA.S., Integral Transforms in Elasticity Theory Problems. Nauka, Leningrad, 1967.
2. BATEMAN H. and ERDELYI A., Higher Transcendental Functions /Russian translation/, Nauka, Moscow, 1965.
3. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series, and Products. Nauka, Moscow, 1971.
4. ABRAMOWITZ M. and STEGUN I., Eds. Handbook on Special Functions with Formulas, Graphs, and Mathematical Tables /Russian translation/, Nauka, Moscow, 1979.
5. POPOV G.YA., Elastic Stress Concentration around Stamps, Slits, Thin Inclusions and Reinforcements. Nauka, Moscow, 1982.
6. SAVRUK M.P., Two-dimensional Elasticity Problems for Bodies with Cracks. Naukova Dumka, Kiev, 1981.
7. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Non-classical Mixed Problems of Elasticity Theory. Nauka, Moscow, 1974.
8. PARTON V.Z. and PERLIN P.I., Methods of Mathematical Elasticity Theory.Nauka, Moscow, 1981.
9. BATEMAN H. and ERDELYI A., Tables of Integral Transforms, Vol.2, /Russian translation/, Nauka, Moscow, 1970.
10. PARTON V.Z. and PERLIN P.I., Integral Equations of Elasticity Theory. Nauka, Moscow, 1977.

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